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Splitting of Linear Operators Applied to the Stationary Navier–Stokes Equation

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Discretization of the Navier–Stokes equation leads to the problem of solving $(M - N(u))u = f$, where M is positive definite symmetric and $N(v)$ skew symmetric. Recent results in linear iterative methods have developed algorithms for the solution of $(M - N)u = f$. We apply the linear methods and a process of linearization to the Navier–Stokes equation on the computer.

1. INTRODUCTION

The solution of linear equations of the form

$$(M - N)\,u = f \tag{1.1}$$

for M positive definite dymmetric and N skew symmetric, has been extensively studied (Concus and Golub [3], Widlund [11], Rapoport [8]). Iterative methods have been developed which require no storage of matrices; however, M, N, and M^{-1} applied to a vector must be easily computable. These methods rely on the specialized positive definite symmetric and skew symmetric structure, and this structure plays a fundamental role in the ensuing results.

The stationary Navier–Stokes equation may be written in the form (Temam [9])

$$(M - N(u))\,u = f, \tag{1.2}$$

where M is linear positive definitive symmetric and $N(v)$ is linear skew symmetric for fixed v. Discretization of the Navier–Stokes equation leaves it in the form (1.2), and it is the solution of these equations which is reported in this research. Equations (1.2) are solved by a process of linearization

$$(M - N(u_n))\,u_{n+1} = f. \tag{1.3}$$

To solve the discrete case we have used a recent development in the finite-element method, that of elements with a divergence-free basis (Thomasset [10]). Of fundamen-

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tal importance in the method is multiplication by M^{-1}. This has been implemented by a Fourier–Toeplitz method (Fischer *et al.* [5]), and a capacitance matrix method (Proskurowski and Widlund [7]).

2. Navier–Stokes Equation

We consider the two-dimensional problem with homogeneous boundary conditions. Let Ω be a bounded domain in R^2 and define

$$W = (H_0^1(\Omega))^2,$$
$$V = \{\mathbf{v} \in W\colon \operatorname{div} \mathbf{v} = 0\}.$$

The problem may be stated in the weak form: Find $\mathbf{u} \in V$, $p \in L^2(\Omega)$ such that for all $\mathbf{w} \in W$,

$$(\operatorname{grad} \mathbf{u}, \operatorname{grad} \mathbf{w}) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[\frac{d}{dx_j}(u_j u_i) + u_j \frac{du_i}{dx_j}\right] w_i \, dx - (p, \operatorname{div} \mathbf{w}) = (f, \mathbf{w}). \tag{2.1}$$

R is the Reynolds number and (,) is the L^2 inner product. The problem may be equivalently stated (Temam [9]):
Find $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$

$$(\operatorname{grad} \mathbf{u}, \operatorname{grad} \mathbf{v}) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[\frac{d}{dx_j}(u_j u_i) + u_j \frac{du_i}{dx_j}\right] v_i \, dx = (f, \mathbf{v}). \tag{2.2}$$

Equation (2.2) is in the weak form of Eq. (1.2),

$$(M - N(\mathbf{u}))\, \mathbf{u}, \mathbf{v}) = (f, \mathbf{v}). \tag{2.3}$$

For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, the trilinear term

$$(N(\mathbf{u})\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega \left[\frac{d}{dx_j}(u_j v_i) + u_j \frac{dv_i}{dx_j}\right] w_i \, dx$$

is skew, $(N(\mathbf{u})\, \mathbf{v}, \mathbf{v}) = 0$, as may easily be seen by integrating by parts to obtain

$$(N(\mathbf{u})\mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega \left[u_j \frac{dv_i}{dv_j} w_i - u_j \frac{dw_i}{dx_j} v_i\right] dx.$$

This yields an equation equivalent to (2.2),

$$(\operatorname{grad} \mathbf{u}, \operatorname{grad} \mathbf{v}) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[u_j \frac{du_i}{dx_j} v_i - u_j \frac{dv_i}{dx_j} u_i\right] dx = (f, \mathbf{v}). \tag{2.4}$$

Equations (2.2) and (2.4) are in the form of (2.3), and may be solved by the linearized procedure

$$(M - N(\mathbf{u}_n))\,\mathbf{u}_{n+1}, \mathbf{v}) = (f, \mathbf{v}). \tag{2.5}$$

More generally, Crouzeix [4] analyzes

$$(A(\mathbf{u})\,\mathbf{u}, \mathbf{v}) = (f, \mathbf{v}) \tag{2.6}$$

for

$$(A(\mathbf{v})\,\mathbf{u}, \mathbf{u}) \geqslant \alpha \| \mathbf{u} \|^2. \tag{2.7}$$

The Navier–Stokes equation is the case that

$$A(\mathbf{v}) = M - N(\mathbf{v}).$$

The analysis of this method and its application to the Navier-Stokes equation are due to Crouzeix. Since this method is of central importance to our work, we discuss it, essentially following Crouzeix [4].

THEOREM 1. *Let H be a Hilbert space, $V \subset H$ having a countable base, V' the dual of V with the dual norm $\| \|^*$. Let $A(\mathbf{v}) \in L(V, V')$ for $\mathbf{v} \in V$ be such that if $\mathbf{v}_n$ is a sequence in V converging weakly to $\mathbf{v}$ then $A(\mathbf{v}_n)\,\mathbf{v}_n$ converges weakly to $A(\mathbf{v})\,\mathbf{v}$ in V'. Also assume that for $\mathbf{u}, \mathbf{v} \in V$, (2.7) holds, and also*

$$\| A(\mathbf{u}) - A(\mathbf{v}) \|_{L(V,V')} \leqslant \phi(\| \mathbf{u} \|, \| \mathbf{v} \|)\, \| \mathbf{u} - \mathbf{v} \|, \tag{2.8}$$

$$\| f \|^* \, \phi\left(\frac{\| f \|^*}{\alpha}, \frac{\| f \|^*}{\alpha}\right) < \alpha^2, \tag{2.9}$$

where $\alpha > 0$ and $\phi : R^{+^2} \to R^+$ is nondecreasing in each variable. Then a unique solution to (2.6) exists and may be obtained by the iteration

$$\mathbf{u}_0 = 0,$$
$$(A(\mathbf{u}_n)\,\mathbf{u}_{n+1}, \mathbf{v}) = (f, \mathbf{v}). \tag{2.10}$$

Before the theorem is proved two lemmas need to be established.

LEMMA 1. *Let P be a continuous function from R^n into R^n such that for all ξ, $|\xi| = \rho$,*

$$(P(\xi), \xi) \geqslant 0. \tag{2.11}$$

Then there exists ξ, $|\xi| \leqslant \rho$, such that ξ is a root of P.

Proof. Assume the contrary, that P has no root in $K = \{\xi : |\xi| \leqslant \rho\}$. The map

$\xi \to -\rho P(\xi)/| P(\xi)|$ maps K into K continuously. By the Brouwer fixed-point theorem there exists a ξ_0 such that

$$\xi_0 = -\rho P(\xi_0)/| P(\xi_0)| . \tag{2.12}$$

Taking the Euclidean inner product of (2.12) with $P(\xi_0)$ we obtain

$$(P(\xi_0), \xi_0) = -\rho \mid P(\epsilon_0)| < 0,$$

a contradiction to (2.16). Q.E.D.

LEMMA 2. *Let A and ϕ be as in* (2.7) *and* (2.8). *Then*

$$\| A^{-1}(\mathbf{u}) - A^{-1}(\mathbf{v})\| \leqslant \frac{\phi(\| \mathbf{u} \|, \| \mathbf{v} \|) \| \mathbf{u} - \mathbf{v} \|}{\alpha^2} .$$

Proof.

$$\begin{aligned} \| A^{-1}(\mathbf{u}) - A^{-1}(\mathbf{v})\| &= \| A^{-1}(\mathbf{u})[A(\mathbf{v}) - A(\mathbf{u})] A^{-1}(\mathbf{v})\| \\ &\leqslant \| A^{-1}(\mathbf{u})\| \| A^{-1}(\mathbf{v})\| \| A(\mathbf{v}) - A(\mathbf{u})\| . \end{aligned}$$

By applying (2.7) and (2.8) the lemma is proved. Q.E.D.

Proof of Theorem 1. First we will use Lemma 1 to prove existence. Let $\phi_1, \phi_2, \ldots,$ be an orthonormal base for V. Let V_n be the Hilbert space spanned by $\phi_1, \phi_2, \ldots, \phi_n$ with the inner product $(\mathbf{x}, \mathbf{y})_{V_n} = (\mathbf{x}, \mathbf{y})_V$. Define $P_n(\mathbf{v}) \in V_n$ for $\mathbf{v} \in V$ by

$$P_n(\mathbf{v}) = \sum_{i=1}^{n} \beta_i \phi_i ,$$

$$\beta_i = (A(\mathbf{v})\, \mathbf{v} - f, \phi_i).$$

We have for $\mathbf{v} \in V_n$ that

$$(P_n(\mathbf{v}), \mathbf{v}) \geqslant (\alpha \| \mathbf{v} \| - \| f \|^*) \| \mathbf{v} \| .$$

Hence for $\| \mathbf{v} \| = \| f \|^*/\alpha = \rho$, (2.11) holds. On V_n, P_n may be viewed as a continuous function from R^n into R^n. By Lemma 1, P_n has a root $\mathbf{w}_n$ in V_n.

$$(A(\mathbf{w}_n)\, \mathbf{w}_n , \phi_i) = (f, \phi_i), \qquad i \leqslant n,$$

with $\| \mathbf{w}_n \| \leqslant \| f \|^*/\alpha$. Every bounded sequence has a weakly convergent subsequence, also denoted by $\mathbf{w}_n \to \mathbf{u}$, with

$$(A(\mathbf{w}_n)\, \mathbf{w}_n , \phi_i) = (f, \phi_i), \qquad i \leqslant n.$$

Letting $n \to \infty$

$$(A(\mathbf{u})\,\mathbf{u}, \phi_i) = (f, \phi_i).$$

Hence in the weak sense

$$A(\mathbf{u})\,\mathbf{u} = f.$$

Existence is now proved and we turn to the iterative procedure. Let $\mathbf{u}$ be a solution of (2.6). Define the sequence $\mathbf{u}_n$ by $\mathbf{u}_0 = 0$ and (2.10). Then by Lemma 2, the a priori bound $\|\mathbf{u}\| \, \|\mathbf{u}_n\| \leqslant \|f\|^*/\alpha$, and assumption (2.9)

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_{n+1}\| &= \|A^{-1}(\mathbf{u})f - A^{-1}(\mathbf{u}_n)f\| \\ &\leqslant \frac{\|f\|^* \, \phi(\|\mathbf{u}\|, \|\mathbf{u}_n\|) \, \|\mathbf{u} - \mathbf{u}_n\|}{\alpha^2} \\ &< \|\mathbf{u} - \mathbf{u}_n\|.\end{aligned}$$

The a priori estimate follows by letting $\mathbf{v}$ equal $\mathbf{u}$ and $\mathbf{u}_{n+1}$ in (2.6) and (2.10), respectively. A similar argument establishes uniqueness. Q.E.D.

3. Discrete Navier–Stokes Equation

We now examine methods for the numerical solution of the Navier–Stokes equation. It is natural to first consider a finite-difference approximation to (2.1)

$$-\Delta_h \mathbf{u} + 0.5RN_h(\mathbf{u}, \mathbf{u}) + \operatorname{grad}_h p = f, \tag{3.1}$$

$$N_h(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} u_1 D_{0x} v_1 + D_{0x}(u_1 v_1) + u_2 D_{0y} v_1 + D_{0y}(u_2 v_1) \\ u_1 D_{0x} v_2 + D_{0x}(u_1 v_2) + u_2 D_{0y} v_2 + D_{0y}(u_2 v_2) \end{bmatrix},$$

$$\operatorname{div}_h \mathbf{u} = 0.$$

Δ_h is the five-point Laplacian, grad_h and div_h are centered difference approximations to grad and div, and D_{0x} and D_{0y} are centered difference approximations to d/dx and d/dy. The positive definite symmetric and skew nonlinear form of the differential equations is preserved in the difference equations. Assuming homogeneous boundary conditions for $\mathbf{w}$,

$$(N_h(\mathbf{v}, \mathbf{w}), \mathbf{w}) = 0$$

for arbitrary $\mathbf{v}$, and $N_h(\mathbf{v}, \mathbf{w})$ is linear in each of $\mathbf{v}$ and $\mathbf{w}$.

The linearized equations corresponding to (3.1) are

$$\begin{aligned}-\Delta_h \mathbf{u}_{n+1} + 0.5RN_h(\mathbf{u}_n, \mathbf{u}_{n+1}) + \operatorname{grad} p_{n+1} &= f, \\ \operatorname{div}_h \mathbf{u}_{n+1} &= 0.\end{aligned} \tag{3.2}$$

Equations (3.2) are not of the form (1.1) due to the appearance of the pressure p_{n+1} and the continuity equation $\operatorname{div}_h \mathbf{u}_{n+1} = 0$.

Two methods, (3.3) and (3.4), discussed by Crouzeix [4] and Temam [9], respectively, allow use of methods for the iterative solution of (1.1).

$$\begin{aligned} -\Delta_h \mathbf{u}_{n+1} + 0.5RN_h(\mathbf{u}_n, \mathbf{u}_{n+1}) + \operatorname{grad}_h p_n &= f, \\ p_{n+1} = p_n - \rho \operatorname{div}_h \mathbf{u}_{n+1}. \end{aligned} \tag{3.3}$$

$$\begin{aligned} -\Delta_h \mathbf{u}_{n+1} + 0.5RN_h(\mathbf{u}_{n+1}, \mathbf{u}_{n+1}) + \operatorname{grad}_h p_n &= f, \\ p_{n+1} = p_n - \rho \operatorname{div}_h \mathbf{u}_{n+1}. \end{aligned} \tag{3.4}$$

ρ is a small positive constant. Fortin *et al.* [6] report good computation results for a method close to (3.4), differing in the discretization of the nonlinear term.

The major difficulty is solving the difference equation (3.1) or its linearized version (3.2) simultaneously with the continuity equation. The usual finite-element equations would give rise to the same problem. For this reason we have used finite elements that satisfy

$$\operatorname{div}_h \mathbf{v}_h = \operatorname{div} \mathbf{v}_h = 0. \tag{3.5}$$

Our reference for the construction of these elements is Thomasset [10].

We will now describe these elements, in particular on the mesh illustrated in Fig. 1. Define the scalar shape function $v_{hM}(x, y)$ on a given triangle as a linear function $ax + by + c$ that takes the value 0 at the two nodes (midpoints of sides) other than M and takes the value 1 at M; see Fig. 2. Define two types of elements, rotations $\mathbf{v}_{hs}(x, y)$ and translations $\mathbf{v}_{hM}(x, y)$. A rotation $\mathbf{v}_{hS}(x, y)$ has support on six adjacent triangles as in Fig. 3. On the triangle with vertices S, S', and S''

$$\mathbf{v}_{hS}(x, y) = \frac{v_{hM'}(x, y)\, \mathbf{n}'}{|SS'|} + \frac{v_{hM''}(x, y)\, \mathbf{n}''}{|SS''|},$$

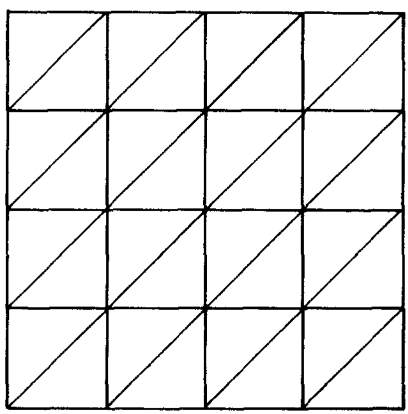

FIGURE 1

where $\mathbf{n}'$, $\mathbf{n}''$ are unit normals in counterclockwise orientation at M', M'', respectively. A translation $\mathbf{v}_{hM}(x, y)$ has support on two adjacent triangles. An example is shown in Fig. 4. On either triangle

$$\mathbf{v}_{hM}(x, y) = \frac{\mathbf{SS}'}{|SS'|^2} v_{hM}(x, y).$$

It follows from direct calculation that these functions are divergence free.

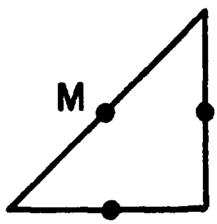

FIGURE 2

We have described these elements on a coarse mesh in the unit square, although they may also be used for other two-dimensional domains. Thomasset [10] has shown that in a simply connected domain the divergence-free elements span the space of 2-vectors with linear entries having zero divergence.

Defining V_h to be the aforementioned space, we may write the discrete versions of (2.2) and (2.4) as

$$(\operatorname{grad} \mathbf{u}_h, \operatorname{grad} \mathbf{v}_h) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[\frac{d}{dx_j}(u_{hj}u_{hi}) + u_{hi}\frac{du_{hi}}{dx_j}\right] v_{hi}\,dx = (f, \mathbf{v}_h) \tag{3.6}$$

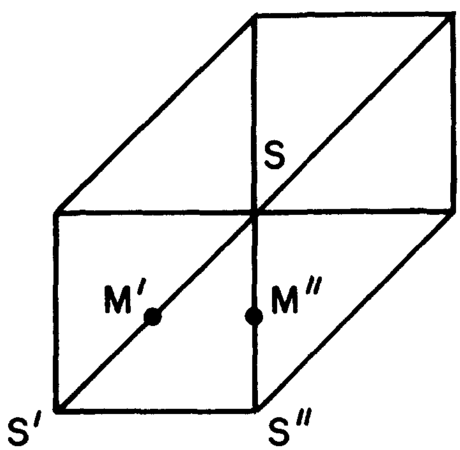

FIGURE 3

and

$$(\text{grad } \mathbf{u}_h, \text{grad } \mathbf{v}_h) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[u_{hi} \frac{du_{hi}}{dx_j} v_{hi} - u_{hj} \frac{dv_{hi}}{dx_j} u_{hi}\right] dx = (f, \mathbf{v}_h). \tag{3.7}$$

Equations (3.5) and (3.6) are not identical, as the elements $\mathbf{v}_h$ are nonconforming ($\mathbf{v}_h \notin (H_0^1(\Omega))^2$). Equation (3.7) is of the form (2.3), positive definite symmetric plus skew nonlinear. The same is also true for Eq. (3.6), to within a small truncation error. These equations may be solved using the linearized method (2.5) and the linear iterative methods for (1.1). The discrete linearized equations are

$$(\text{grad } \mathbf{u}_h, \text{grad } \mathbf{v}_h) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[\frac{d}{dx_j} (u_{hj}^n u_{hi}^{n+1}) + u_{hj}^n \frac{du_{hi}^{n+1}}{dx_j}\right] v_{hi}\, dx = (f, \mathbf{v}_h) \tag{3.8}$$

and

$$(\text{grad } \mathbf{u}_h, \text{grad } \mathbf{v}_h) + 0.5R \sum_{i,j=1}^{2} \int_\Omega \left[u_{hj}^n \frac{du_{hi}^{n+1}}{dx_j} v_{hi} - u_{hj}^n \frac{dv_{hi}}{dx_j} u_{hi}^{n+1}\right] dx = (f, \mathbf{v}_h).$$

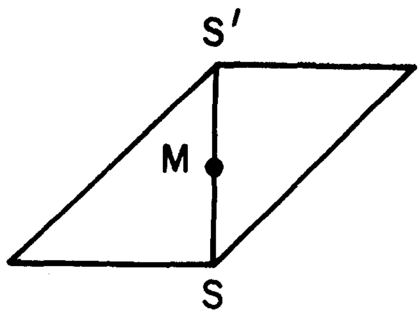

FIGURE 4

4. NUMERICAL RESULTS

We attempt to solve a standard problem, the driven square cavity (Burgraff [1], Chorin [2], Fortin *et al.* [6], Thomasset [10]). Ω is the unit square, and the boundary conditions are

$$\begin{aligned}
&u_1 = -1,\ u_2 = 0, && 0 \leqslant x \leqslant 1, && y = 1,\\
&u_1 = u_2 = 0, && 0 \leqslant x \leqslant 1, && y = 0,\\
&u_1 = u_2 = 0, && 0 \leqslant y = 1, && x = 0 \text{ and } x = 1.
\end{aligned}$$

To apply the linear methods to (3.6) and (3.7) the stiffness matrix for the linear term must be inverted. A direct method rather than an iterative method is preferred, using as little storage as possible. For this reason we reject Cholesky decomposition and use a Fourier–Toeplitz method related to the work of Fisher *et al.* [5]. The first step is to invert the stiffness matrix on the unit square periodic in the x-direction. For each mesh point (i, j) we associate four elements $\alpha(i, j)$, $\beta(i, j)$, $\gamma(i, j)$, and $\delta(i, j)$ and a corresponding vector of coefficients $X(i, j) = \{X_k(i, j)\}_1^4$, as in Fig. 5. Order the (i, j) lexicographically. The stiffness matrix represents difference equations periodic in the x-direction, which may be inverted using the fast Fourier transform. It is possible to show more generally that difference equations with constant coefficients may be inverted using the fast Fourier transform. Neglect for the moment the presence of boundaries at $y = 0$ and $y = 1$. Write the equations as

$$\begin{aligned} &A_{-1,-1}X(i-1, j-1) + A_{0,-1}X(i, j-1) + A_{1,-1}X(i+1, j-1) \\ &\quad + A_{-1,0}X(i-1, j) + A_{0,0}X(i, j) + A_{1,0}X(i+1, j) \\ &\quad + A_{-1,1}X(i-1, j+1) + A_{0,1}X(i, j+1) \\ &\quad + A_{1,1}X(i+1, j+1) = F(i, j), \qquad i, j = 0,..., N-1. \end{aligned}$$

$A_{l,m}$ is a 4×4 matrix equal to

$$\begin{bmatrix} -\langle\alpha(i+l, j+m), \alpha(i, j)\rangle - \langle\alpha(i+l, j+m), \beta(i, j)\rangle \\ \qquad - \langle\alpha(i+l, j+m), \gamma(i, j)\rangle - \langle\alpha(i+l, j+m), \delta(i, j)\rangle \\ -\langle\beta(i+l, j+m), \alpha(i, j)\rangle - \langle\beta(i+l, j+m), \beta(i, j)\rangle \\ \qquad - \langle\beta(i+l, j+m), \gamma(i, j)\rangle - \langle\beta(i+l, j+m), \delta(i, j)\rangle \\ -\langle\gamma(i+l, j+m), \alpha(i, j)\rangle - \langle\gamma(i+l, j+m), \beta(i, j)\rangle \\ \qquad - \langle\gamma(i+l, j+m), \gamma(i, j)\rangle - \langle\gamma(i+l, j+m), \delta(i, j)\rangle \\ -\langle\delta(i+l, j+m), \alpha(i, j)\rangle - \langle\delta(i+l, j+m), \beta(i, j)\rangle \\ \qquad - \langle\delta(i+l, j+m), \gamma(i, j)\rangle - \langle\delta(i+l, j+m), \delta(i, j)\rangle \end{bmatrix}$$

$$\langle\cdot, \cdot\rangle = (\text{grad}\ \cdot, \text{grad}\ \cdot)_{L^2}.$$

Expand X and F in a discrete Fourier series in the x-direction.

$$\begin{aligned} X(i, j) &= \sum_{\omega=0}^{N-1} e^{i\omega 2\pi ih}\hat{X}(\omega, j), \\ F(i, j) &= \sum_{\omega=0}^{N-1} e^{i\omega 2\pi ih}\hat{F}(\omega, j). \end{aligned} \tag{4.2}$$

The discrete Fourier series is unique, and the inversion formula to (4.2) is

$$\hat{X}(\omega, j) = \frac{1}{N}\sum_{i=0}^{N-1} e^{-i\omega 2\pi ih}X(i, j).$$

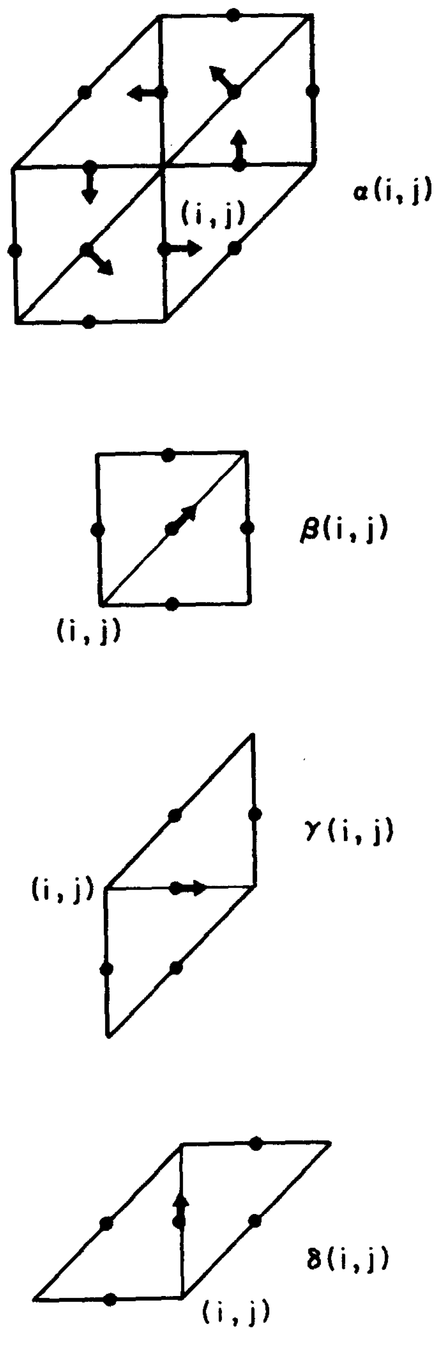

FIGURE 5

Now write the difference equations (4.1) as

$$\sum_{\omega=0}^{N-1} (A_{-1,-1}e^{-i\omega 2\pi h} + A_{0,-1} + A_{1,-1}e^{i\omega 2\pi h})\, e^{i\omega 2\pi ih}\hat{X}(\omega, j-1)$$

$$+ \sum_{\omega=0}^{N-1} (A_{-1,0}e^{-i\omega 2\pi h} + A_{0,0} + A_{1,0}e^{i\omega 2\pi h})\, e^{i\omega 2\pi ih}\hat{X}(\omega, j)$$

$$+ \sum_{\omega=0}^{N-1} (A_{-1,1}e^{-i\omega 2\pi h} + A_{0,1} + A_{1,1}e^{i\omega 2\pi h})\, e^{i\omega 2\pi ih}\hat{X}(\omega, j+1)$$

$$= \sum_{\omega=0}^{N-1} e^{i\omega 2\pi ih}\hat{F}(\omega, j), \qquad i, j = 0,\ldots, N-1. \tag{4.3}$$

Define

$$B(\omega) = A_{-1,-1}e^{-i\omega 2\pi h} + A_{0,-1} + A_{1,-1}e^{i\omega 2\pi h},$$
$$M(\omega) = A_{-1,0}e^{-i\omega 2\pi h} + A_{0,0} + A_{1,0}e^{i\omega 2\pi h}.$$

Using the uniqueness of the discrete Fourier transform and the fact that $A_{l,m} = A^T_{-l,-m}$, (4.3) becomes

$$B(\omega)\,\hat{X}(\omega, j-1) + M(\omega)\,\hat{X}(\omega, j) + B^*(\omega)\,\hat{X}(\omega, j+1) = \hat{F}(\omega, j),$$
$$\omega, j = 0,\ldots, N-1. \tag{4.4}$$

These difference equations are linked only in the y-direction. Thus we have reduced (4.1) to a block-tridiagonal system of linear equations. The Dirichlet boundary conditions at $y = 0$ and $y = 1$ present only a programming detail. These equations were solved using complex block-Gaussian elimination. The storage requirements are $16N^2$ complex words for the factorization. The pivotal elements are on the diagonal, and so no storage is needed for pivotal indices. For completeness we exhibit $M(\omega)$ and $B(\omega)$,

$$M(\omega) = \begin{bmatrix} 16 - 8\cos(2\pi\omega h) & 4e^{-2\pi i\omega h} & 1 - e^{-2\pi i\omega h} & -1 - e^{2\pi i\omega h} \\ 4e^{2\pi i\omega h} & 16 & -2 & -2 - 2e^{2\pi i\omega h} \\ 1 - e^{2\pi i\omega h} & -2 & 4 & 0 \\ -1 - e^{-2\pi i\omega h} & -2 - 2e^{-2\pi i\omega h} & 0 & 4 \end{bmatrix},$$

$$B(\omega) = \begin{bmatrix} -4 & -4 & -e^{-2\pi i\omega h} & 1 + e^{-2\pi i\omega h} \\ 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now that the stiffness matrix representing the Laplacian in the unit square for the x-direction periodic is inverted, we may invert the stiffness matrix for the Dirichlet problem by using the capacitance matrix method. The capitance matrix method (Proskirowski and Widlund [7]), is a tool from linear algebra for the simultaneous inversion of matrices which are equal in most of their entries, or which are embedded one in the other. We now give a precise description of the method in the latter case.

Let A be an $(n-p) \times (n-p)$ invertible matrix and

$$B = \begin{bmatrix} A & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

an $(n \times n)$ invertible matrix. For computational purposes it is desirable that p is much smaller than n and that the inverse of B times a vector is easily computed. B_{21}, B_{12}, and B_{22} are $p \times (n-p)$, $(n-p) \times p$, and $p \times p$ matrices, respectively. Define the $p \times p$ capacitance matrix C by

$$C = (O_{p\times(n-p)} I_{p\times p})\, B^{-1} \begin{bmatrix} O_{(n-p)\times p} \\ I_{p\times p} \end{bmatrix}.$$

To solve

$$Au = v$$

first the capacitance matrix is calculated and factored by Gaussian elimination. Then

$$\bar{u} = (O_{p\times(n-p)} I_{p\times p})\, B^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix}$$

is computed and

$$C\beta = -\bar{u}$$

is solved. The solution u is given by

$$u = (I_{(n-p)\times(n-p)} O_{(n-p)\times p})\, B^{-1} \begin{bmatrix} v \\ \beta \end{bmatrix}.$$

That the capacitance matrix is invertible and u is the solution are easily verified.

Using the capacitance matrix method the Dirichlet problem may be solved using the Fourier method for the periodic problem. A is simply the stiffness matrix for the Dirichlet problem and B the stiffness matrix for the periodic problem. The additional equations of B are those corresponding to elements which no longer lie in the solution space. They are the $\alpha(i, 0)$ and $\delta(i, 0)$ (refer to Fig. 5). Since B is positive definite symmetric the capacitance matrix is positive definite symmetric. The storage requirements for the code are $O(N^2)$; however, at the expense of increased computation times this figure could be reduced to $O(N)$.

The proposed numerical method was successful in solving (3.6) and (3.7). It was found computationally more efficient to solve the corresponding linearized equations (3.8) and (3.9) until the norm of the residual of Eqs. (3.8) and (3.9) was less than a given epsilon (e.g., $\epsilon = 10^{-3}$), rather than solving (3.8) and (3.9) to within machine accuracy. The finest mesh on which a solution was calculated was for N = number of divisions along coordinate axes = 16. The mesh illustrated in Fig. 1 corresponds to $N = 4$.

The solutions of (3.6) and (3.7) at $R = 100$ were virtually identical, and differed at $R = 250$. We present a plot of the solution of (3.6) at $R = 100$ in Fig. 6 and a graph of the velocity profile along the centerline $x = 0.5$ of the same in Fig. 7. These may be compared with similar data in the references mentioned at the beginning of this section for verification that the method is viable for small Reynolds numbers. To make the method successful at large Reynolds numbers an exceedingly fine mesh

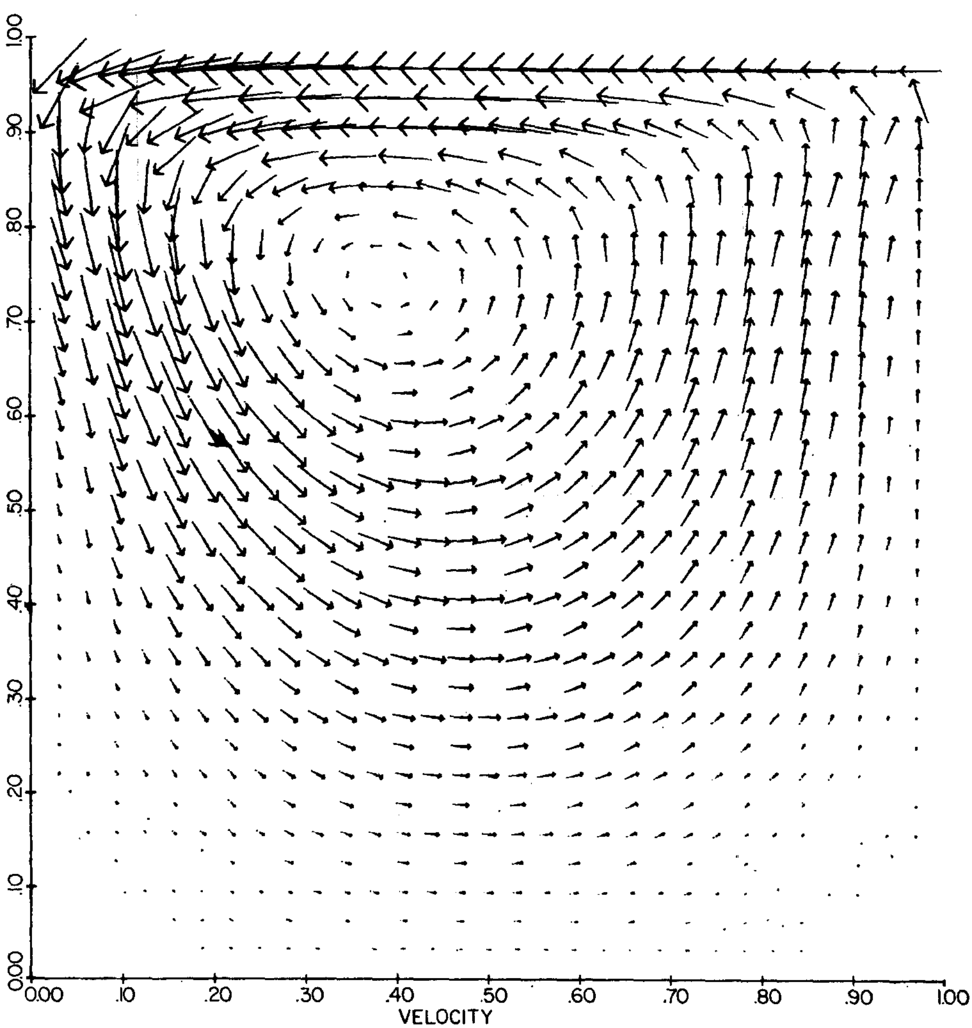

FIGURE 6

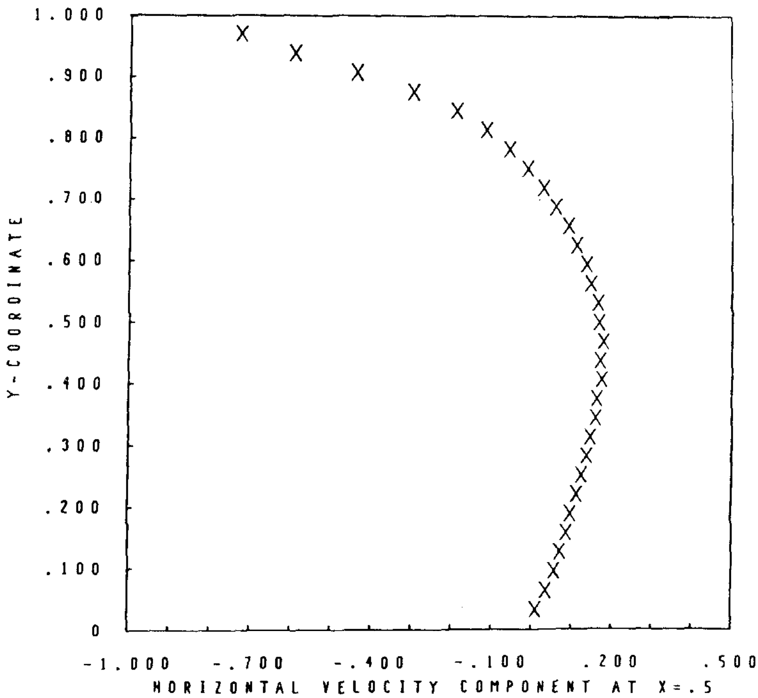

FIGURE 7

would have to be used, requiring additional software problems. It should also be mentioned that the method is adaptable to nonrectangular domains. In Table I we have included computation times for the illustrated run at the Courant Institute of Mathematical Sciences CDC 6600.

TABLE I

Reynolds number	CPU seconds execution time	$\dfrac{\|\text{Residual}\|}{\|\text{Solution}\|}$	Number of linearizations
100	167	0.11×10^{-3}	10
100	239	0.26×10^{-4}	20

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